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VIBRATION CONTROL FOR THE PRIMARY RESONANCE OF A CANTILEVER BEAM BY A TIME DELAY STATE FEEDBACK

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The primary resonance of a cantilever beam under state feedback control with a time delay is investigated. By means of the asymptotic perturbation method, two slow-flow equations on the amplitude and phase of the oscillator are obtained and external excitation–response and frequency–response curves are shown. Vibration control and high-amplitude response suppression can be performed with appropriate time delay and feedback gains. Moreover, energy considerations are used in order to investigate existence and characteristics of quasiperiodic modulated motion for the cantilever beam. It can be demonstrated that appropriate choices for the feedback gains and the time delay can exclude the possibility of modulated motion and reduce the amplitude peak of the primary resonance. Analytical results are verified with numerical simulations.

1. INTRODUCTION

Over the last few years, numerous papers have been dedicated to the control of resonantly forced systems in various engineering fields. In passive vibration absorbers, a physical device is connected to the primary structure, while in the case of active absorbers, the device is replaced by a control system of sensors, actuators and filters. Active control of mechanical and structural vibrations is superior to passive control, because the former is more flexible in many aspects. For example, Oueni *et al.* [1] have considered a non-linear active vibration absorber coupled with the plant through user-defined cubic non-linearities. If the plant is excited at primary resonance and the absorber frequency is equal to the plant natural frequency, they demonstrated that when the forcing amplitude increases beyond a certain threshold, high-amplitude vibrations are suppressed because the response amplitude of the plant remains constant, while the response amplitude of the absorber increases. Oueni *et al.* [2] investigated the saturation phenomenon in devising an active vibration suppression technique. A plant was coupled with a second order absorber through a user-defined quadratic feedback control law. They demonstrated that, by tuning the natural frequency of the absorber to one-half the excitation frequency, effective vibration suppression is possible.

However, unavoidable time delays in controllers and actuators give rise to complicated dynamics and can produce instability of the controlled systems. Moiola *et al.* [3] considered Hopf bifurcations in non-linear feedback systems with time delay. Periodically forced non-linear systems under delay control have been investigated by Plaut and Hsieh [4] in the case of non-linear structural vibrations with a time delay in damping. Hu *et al.* [5]

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considered primary resonance and the $\frac{1}{3}$ subharmonic resonance of a forced Duffing oscillator with time delay state feedback. Using the multiple scales method [6, 7], they demonstrated that appropriate choices of the feedback gains and the time delay are possible for better vibration control. Tsuda *et al.* [8] investigated a non-linear vibrating system with some time delay and numerically detected chaotic behaviour.

Hy and Zh [9] considered controlled mechanical systems with time delays and, in particular, primary resonance and subharmonic resonance of a harmonically forced Duffing oscillator with time delay. Stabilization of periodic motion and applications to active chassis of ground vehicles are discussed.

In previous papers, the response of a parametrically or externally excited van der Pol oscillator has been investigated, and it has been shown that vibration control and quasiperiodic motion suppression are possible for appropriate choices of time delay and feedback gains [10, 11].

In this paper, a cantilever beam whose response is governed by a non-linear partial-differential equation is considered. If we focus on a mode that is not involved in an internal resonance with any of the other modes, then application of a single-mode discretization scheme (see reference [12]) yields the non-linear difference–differential equation

$$\ddot{X}(t) + \omega^2 X(t) + a\dot{X}(t) + bX^3(t) + cX^2(t)\ddot{X}(t) + dX(t)\dot{X}^2(t) - 2f\cos(\Omega t) + AX(t-T) + B\dot{X}(t-T) = 0, \qquad (1)$$

where dot denotes differentiation with respect to time, ω is the natural frequency, a is a damping coefficient, b is the curvature non-linearity coefficient, c and d are the inertia non-linearities coefficients, f is the forcing amplitude and the external excitation frequency is $\Omega \approx \omega$ (primary resonance). A and B are the feedback gains and T is the time delay.

The paper is arranged as follows. In section 2, using the asymptotic perturbation (AP) method [10, 11], a lowest order approximate solution of the non-linear oscillator (1) is constructed. The AP method is based on large temporal rescalings and balancing of harmonic terms with a simple iteration, and then can be considered as an attempt to link the most useful characteristics of harmonic balance and multiple scale methods. Only a slow time scale is used and harmonics are introduced for the fast time scale. However, for the first order approximate solution, results are identical to those obtainable with the other perturbation methods. Obviously, there may be other solutions, for example large-amplitude quasiperiodic motion or chaotic behaviour, that the slow flow equations do not describe.

It is demonstrated that the dynamics of the controlled and uncontrolled oscillators are essentially the same when an appropriate redefinition of the damping coefficient and the detuning parameter is accomplished.

In section 3a bifurcation analysis is performed and external excitation–response and frequency–response curves are shown for the uncontrolled system, the controlled system without time delay, and those with time delays corresponding to the minimum and maximum value of the equivalent damping. It is found that the amplitude peak of the primary resonance can be reduced by means of a correct choice of the time delay and the feedback gains.

In section 4a global analysis of the slow-flow equations is performed and energy considerations are used in order to study existence and characteristics of limit cycles of the slow-flow equations. A limit cycle corresponds to a two-period modulated motion for the externally excited cantilever beam. It is found that no limit cycles (corresponding to a two-period modulated motion for the cantilever beam) exist for the slow-flow equations.

However, the possibility of another type of quasiperiodic motion, characterized by the oscillator phase which is unbounded and grows or diminishes indefinitely, can be demonstrated.

The best choices of the feedback gains and the time delay, from the viewpoint of vibration control, are found. The quasiperiodic motion is suppressed and the amplitude peak of the primary resonance is reduced.

The paper closes with a discussion, along with some conclusions, in section 5.

2. THE AP METHOD AND THE LOWEST ORDER APPROXIMATE SOLUTION

In this section the case of primary resonance is examined and the detuning parameter σ is introduced,

$$\omega = \Omega + \varepsilon\sigma, \tag{2}$$

where ε is a bookkeeping device, which will be set equal to unity in the final analysis. Only the study of small damping, weak non-linearity, weak feedback and soft excitation is considered. Taking into account equation (2), equation (1) can be written in the following form:

$$\ddot{X}(t) + \Omega^2 X(t) + \varepsilon\Big(2\sigma\Omega X(t) + \varepsilon\sigma^2 X(t) + a\dot{X}(t) + bX^3(t) + cX^2(t)\ddot{X}(t) + dX(t)\dot{X}^2(t)X\Big) \\ +\varepsilon\big(AX(t-T) + B\dot{X}(t-T)\big) - 2\varepsilon f\cos(\Omega t) = 0. \tag{3}$$

Modifications induced by non-linearities and parametric resonance are best described by the slow temporal scale

$$\tau = \varepsilon\, t, \tag{4}$$

which is associated with modulations in the amplitude and the phase of the solution.

The approximate solution $X(t)$ is sought in the form of a power series in the expansion parameter ε,

$$X(t) = \sum_{n[odd]=-\infty}^{+\infty} \varepsilon^{\gamma_n}\psi_n(\tau,\varepsilon)\exp(-\mathrm{i}n\Omega t) \tag{5}$$

with $\gamma_n = |n| - 1$. Note that $\psi_n(\tau,\varepsilon) = \psi^*_{-n}(\tau,\varepsilon)$, because $X(t)$ is real. The functions $\psi_n(\tau,\varepsilon)$ depend on the parameter ε and it is supposed that their limit for $\varepsilon \to 0$ exists and is finite.

The solution is then a Fourier expansion in which the coefficients vary slowly in time. The lowest order terms correspond to the harmonic solution of the linear problem. Evolution equations for the amplitudes of the harmonic terms are then derived by substituting the expression of the solution into the original equations and projecting onto each Fourier mode.

Substituting equation (5) into equation (3), considering the coefficients of the most important Fourier mode ($n = 1$) and collecting terms of the same power of ε yields, to order ε,

$$2\mathrm{i}\Omega\frac{\mathrm{d}\psi}{\mathrm{d}\tau} + f - 2\sigma\Omega\psi + \mathrm{i}a\Omega\psi + (3c\Omega^2 - 3b - \mathrm{d}\Omega^2)|\psi|^2\psi + (\mathrm{i}B\Omega - A)\psi\exp(\mathrm{i}\Omega T) = 0, \tag{6}$$

where $\psi = \psi_1$.

To analyze the combined effects of the non-linearity, the primary resonance and the delay control, the polar form

$$\psi = \rho\exp(\mathrm{i}\vartheta) \tag{7}$$

can be substituted into equation (6), in order to separate real and imaginary parts and to obtain

$$\frac{d\rho}{d\tau} + \left(\frac{a}{2} - \frac{A}{2\Omega}\sin(\Omega T) + \frac{B}{2}\cos(\Omega T)\right)\rho - \frac{f}{2\Omega}\sin(\vartheta) = 0, \tag{8}$$

$$\rho\frac{d\vartheta}{d\tau} + \left(\sigma + \frac{B}{2}\sin(\Omega T) + \frac{A}{2\Omega}\cos(\Omega T)\right)\rho + \left(\frac{3b}{2\Omega} - \frac{3\Omega c}{2} + \frac{d\Omega}{2}\right)\rho^3 - \frac{f}{2\Omega}\cos\vartheta = 0. \tag{9}$$

Defining

$$K = \sqrt{\left(\frac{B}{2}\right)^2 + \left(\frac{A}{2\Omega}\right)^2}, \quad \cos\varphi = \frac{B}{2K}, \quad \sin\varphi = \frac{A}{2K\Omega} \tag{10}$$

and substituting equation (10) into equations (8,9) yields

$$\frac{d\rho}{d\tau} + \left(\frac{a}{2} + K\cos(\varphi + \Omega T)\right)\rho - \frac{f}{2\Omega}\sin(\vartheta) = 0, \tag{11}$$

$$\rho\frac{d\vartheta}{d\tau} + (\sigma + K\sin(\Omega T + \varphi))\rho + \left(\frac{3b}{2\Omega} - \frac{3\Omega c}{2} + \frac{d\Omega}{2}\right)\rho^3 - \frac{f}{2\Omega}\cos\vartheta = 0. \tag{12}$$

Equations (11,12) represent a system of first order, autonomous, ordinary differential equations, governing the amplitude and phase of the approximate solution expressed by

$$X(t) = 2\rho\cos(-\Omega t + \vartheta) + O(\varepsilon). \tag{13}$$

From equations (11, 12) it can be seen that the external excitation of the uncontrolled and controlled oscillator are essentially the same when the detuning parameter σ and the damping coefficient A are properly substituted using

$$\sigma \to \sigma + K\sin(\Omega T + \varphi), \quad a \to a + 2K\cos(\Omega T + \varphi). \tag{14a, 14b}$$

On the contrary, the non-linear coefficients b, c and d remain unchanged.

3. STABILITY ANALYSIS AND PRIMARY RESONANCE CONTROL

Setting

$$d\rho/d\tau = d\vartheta/d\tau = 0 \tag{15}$$

in equations (11, 12), the external excitation–response curve for the steady state solution amplitude corresponding to a periodic response of the starting system can be found:

$$f = 2\Omega\rho\sqrt{(\sigma + K\sin(\varphi + \Omega T) + \alpha\rho^2)^2 + \left(\frac{a}{2} + K\cos(\varphi + \Omega T)\right)^2}, \tag{16}$$

where

$$\alpha = \frac{3b}{2\Omega} - \frac{3\Omega c}{2} + \frac{d\Omega}{2}. \tag{17}$$

The stability properties of a constant solution are examined by applying the well-known method of linearization. Small perturbations have been superposed on the steady state solution and the resulting equations are then linearized. Subsequently, the eigenvalues of the corresponding system of first order differential equations with constant coefficients (the Jacobian matrix) are considered. A positive real root indicates an unstable solution, whereas if the real parts of the eigenvalues are all negative then the steady state solution is stable. When the real part of an eigenvalue is zero, bifurcation occurs. A change from complex roots with negative real parts to complex roots with positive real parts could indicate the presence of a supercritical or subcritical Hopf bifurcation. The question of which can actually occur depends on the non-linear terms [13, 14].

The eigenvalues of the Jacobian matrix satisfy the equation

$$\lambda^2 + P\lambda + Q = 0, \tag{18}$$

where

$$P = 2K\cos(\varphi + \Omega T) + a, \tag{19}$$

$$Q = \left(\frac{a}{2} + K\cos(\Omega T + \varphi)\right)^2 + \left(\frac{a}{2} + 3\alpha\rho_0^2 + K\cos(\Omega T + \varphi)\right)\left(\alpha\rho_0^2 + \sigma + K\sin(\Omega T + \varphi)\right). \tag{20}$$

Then the eigenvalues are both negative if

$$P > 0, \quad Q > 0. \tag{21}$$

Results of stability analysis for a typical case are given in Figure 1, together with results obtained by the numerical integration of equation (1), for the uncontrolled system (see curve A), the controlled system without time delay (curve B), and those with time delays

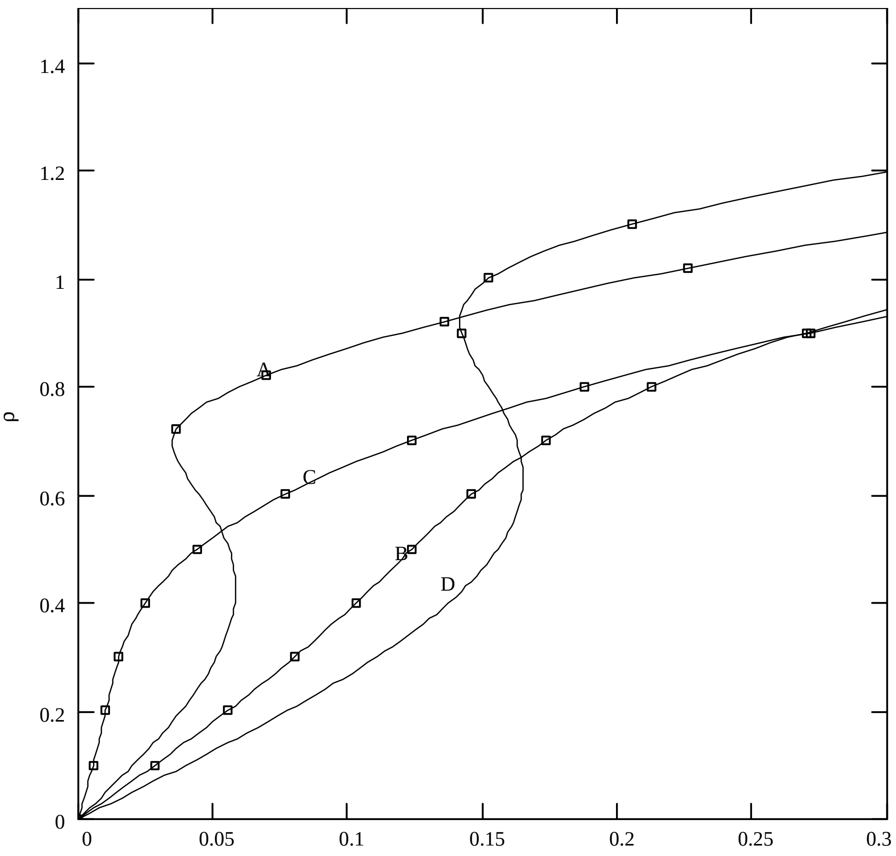

Figure 1. External excitation (f)–response (ρ) curves for the uncontrolled system (curve A), the controlled system with no time delay (curve B), and those with time delay corresponding to the maximum (curve C) and minimum (curve D) value of the equivalent damping. Boxes stand for the numerical solution. ($K = 0{\cdot}1$, $\alpha = 0{\cdot}2$, $\varphi = \pi/12$, $\sigma = -0{\cdot}1$).

corresponding to the maximum (curve C) and the minimum (curve D) equivalent damping, according to equation (14b).

Numerical integration has been performed only for the stable steady state solution curves, and the two curves without boxes correspond to unstable solutions. It can be seen that an appropriate choice of the time delay can reduce the response amplitude and perform an efficient vibration control (case D). Note, however that the time delay can worsen the response (curve C as opposed to curve D in Figure 1) for some range of forcing.

As f increases from zero, only one solution (corresponding to a periodic motion) exists, which is stable in cases B and C and unstable in cases A and D. When f reaches a critical value, there are three possibilities for the curves A and D: two unstable solutions and one stable large-amplitude solution.

On the other hand, in the uncontrolled system (curve A), no steady state solution exists for sufficiently small values of f, and it can be demonstrated (section 4) that in this case a modulated motion can settle down. However, a correct choice of the time delay and the feedback gains can suppress the quasiperiodic motion.

The frequency–response curve is given by

$$\sigma = -K\sin(\varphi + \Omega T) - \alpha\rho^2 \pm \sqrt{\left(\frac{f}{2\Omega\rho}\right)^2 - \left(\frac{a}{2} + K\cos(\varphi + \Omega T)\right)^2} \tag{22}$$

and is shown in Figure 2 for typical cases. Also in this case, the response is strongly influenced by the time delay. The controlled system with maximum equivalent damping (case D) is not adequate and its performance is unsatisfactory, because the response value is substantially unchanged, while for cases B (no time delay) and C (minimum equivalent damping) the response is reduced with respect to the uncontrolled system (curve A).

A better choice for the time delay can be obtained if we consider equation (16) and set $\mathrm{d}\rho/\mathrm{d}T = 0$. The time delay T_0 is then given by the condition

$$\tan(\Omega T_0 + \varphi) = \frac{2(\sigma + \alpha\rho_0^2)}{a}. \tag{23}$$

The two delays T_0 satisfying equation (23) correspond to a minimum and a maximum value for the response. These time delays are not constant but depend on the response value. Numerical integration confirms that the strategy given by the choice of the time delay of equation (23) corresponding to a minimum value of the response is better than the simple choice of the minimum equivalent damping shown in Figures 1 and 2.

4. SUPPRESSION OF THE QUASIPERIODIC MOTION

In many situations, it is desirable to suppress the quasiperiodic motion and reduce the amplitude peak of the primary resonance. It can be demonstrated that this result can be accomplished by appropriate choices of time delay and feedback gains.

Limit cycles of the slow-flow equations (11, 12) correspond to two-period quasiperiodic solutions of the externally excited cantilever beam. A global analysis of the slow-flow equations can be performed and sufficient conditions in order to exclude the existence of limit cycles can be determined. The model system (11, 12) is rewritten in the form

$$\frac{\mathrm{d}R}{\mathrm{d}\tau} - (a + 2K\cos(\varphi + \Omega T))R - \frac{f}{\Omega}\sqrt{R}\sin\vartheta = 0, \tag{24}$$

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}\tau} + \left(\sigma + K\sin(\varphi + \Omega T) + \alpha R - \frac{f}{2\Omega\sqrt{R}}\cos\vartheta\right) = 0, \tag{25}$$

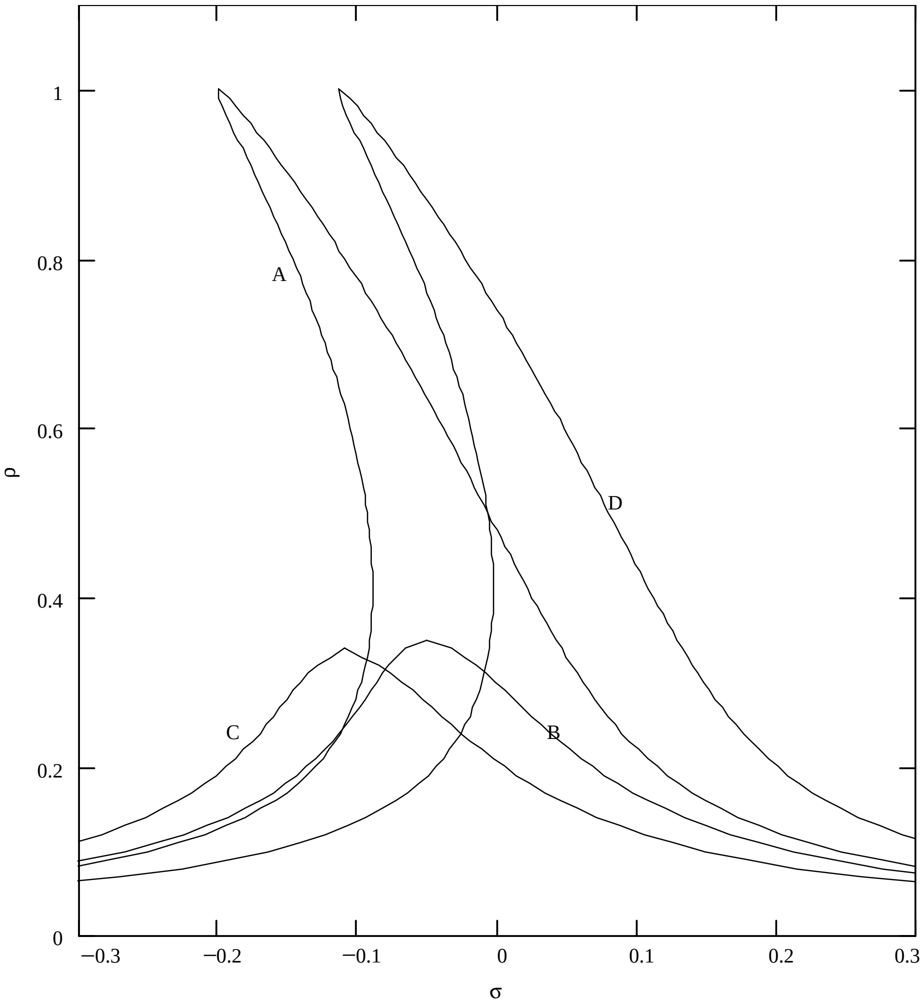

Figure 2. Frequency (σ)–response (ρ) curves for the uncontrolled system (curve A), the controlled system with no time delay (curve B), and those with time delay corresponding to the minimum (curve C) and maximum (curve D) value of the equivalent damping. ($K = 0{\cdot}1$, $\alpha = 0{\cdot}2$, $\varphi = \pi/12, f = 0{\cdot}05$).

where $R = \rho^2$, in order to apply the energy-like function method, which is very useful for the problem of the existence of closed orbits and modulated motion. It is well known that there is no efficacious method to find this function for a given non-linear differential equation, although sometimes one can work backwards.

For equations (24, 25), the energy-like function can be introduced as

$$E(R, \vartheta) = (\sigma + K\sin(\Omega T + \phi))R + \frac{\alpha}{2}R^2 - \frac{f}{\Omega}\sqrt{R}\cos\vartheta \tag{26}$$

If a periodic solution of a generic period $\hat{T}$ exists, then, after one cycle, R and ϑ return to their starting values. Therefore, we find that $\Delta E = 0$ around any closed orbit.

On the other hand, a simple calculation yields

$$\frac{\mathrm{d}E}{\mathrm{d}\tau} = \left(\sigma + K\sin(\Omega T + \phi) + \alpha R - \frac{f}{2\Omega\sqrt{R}}\cos\vartheta\right)\frac{\mathrm{d}R}{\mathrm{d}\tau} + \frac{f\sqrt{R}\sin\vartheta}{\Omega}\frac{\mathrm{d}\vartheta}{\mathrm{d}\tau} \tag{27}$$

and then, using equations (24) and (25),

$$\frac{\mathrm{d}E}{\mathrm{d}\tau} = -(a + 2K\cos(\Omega T + \varphi))R\frac{\mathrm{d}\vartheta}{\mathrm{d}\tau}. \tag{28}$$

From equation (28), it can be derived that, around any closed orbit,

$$\Delta E = \int_0^{\hat{T}} \frac{\mathrm{d}E}{\mathrm{d}\tau}\mathrm{d}\tau = -(a + 2K\cos(\Omega T + \varphi))\int_0^{2\pi} R\mathrm{d}\vartheta. \tag{29}$$

This integral is obviously non-zero, because the integrand function has a constant sign. A contradiction is then obtained and it implies that there is no periodic orbit and no corresponding two-period quasiperiodic motion for the cantilever beam, if ϑ has to increase monotonically from 0 to 2π, as implied by the argument associated with equation (29).

If this last condition is not satisfied, quasiperiodic motion is possible when equation (12) cannot possess fixed points, i.e., when

$$|\sigma + K\sin(\varphi + \Omega T)| \gg \frac{f}{2\Omega\rho} + |\alpha|\rho^2. \tag{30}$$

As a consequence, if the initial conditions satisfy condition (30), then the motion becomes a quasiperiodic modulated motion. In this case the approximate solution for system (11, 12) is

$$\vartheta = \vartheta_0 + \hat{\Omega}\tau, \tag{31}$$

$$\rho(\tau) = \rho_0\exp(-\Lambda\tau) + \frac{f\left[2(\hat{\Omega}\cos\vartheta_0 - \Lambda\sin\vartheta_0)\exp(-\Lambda\tau) + 2\Lambda\sin(\hat{\Omega}\tau + \vartheta_0) - 2\hat{\Omega}\cos(\hat{\Omega}\tau + \vartheta_0)\right]}{4\Omega(\Lambda^2 + \Omega^2)}, \tag{32}$$

where

$$\hat{\Omega} = -\sigma - K\sin(\Omega T + \phi), \quad \Lambda = \frac{a}{2} + K\cos(\Omega T + \phi). \tag{33, 34}$$

If the coefficient $\Lambda > 0$, then the amplitude ρ is slowly modulated and the asymptotic solution is

$$\rho(\tau) = \frac{f\left[\Lambda\sin(\hat{\Omega}\tau + \vartheta_0) - \hat{\Omega}\cos(\hat{\Omega}\tau + \vartheta_0)\right]}{2\Omega(\Lambda^2 + \hat{\Omega}^2)}, \tag{35}$$

corresponding to a quasiperiodic motion for the cantilever beam. The above illustrated analytical considerations have been checked by means of a numerical integration of equations (11, 12).

From the viewpoint of vibration control, it can be seen that for the elimination of the two-period quasiperiodic motion, the feedback gains must be chosen in such a way that the left-hand side of equation (30) is small enough, in order to obtain fixed points for equation (25).

The best choice is obviously

$$K = -\frac{\sigma}{\sin(\varphi + \Omega T)} \tag{36}$$

and from equations (24, 25) the steady state solutions can be easily obtained from the external excitation–response curve:

$$f = \rho\Omega\sqrt{4\alpha^2\rho^4 + (a + 2K\cos(\varphi + \Omega T))^2} \tag{37}$$

and in this case the cantilever beam settles down into a periodic motion.

In order to reduce the amplitude peak, the condition $\mathrm{d}\rho/\mathrm{d}T = 0$ must be satisfied and from equation (37) it can be found that

$$\sin(\Omega T_0 + \varphi) = 0. \tag{38}$$

The delay T_0 corresponds to a minimum or a maximum value for the response, if the second derivative,

$$\left.\frac{\mathrm{d}^2\rho}{\mathrm{d}T^2}\right|_{T=T_0} = \frac{2\rho K\Omega^2\cos(\Omega T_0 + \varphi)(a + 2K\cos(\Omega T_0 + \varphi))}{12\alpha^2\rho^4 + (a + 2K\cos(\Omega T_0 + \varphi))^2}, \tag{39}$$

is, respectively, >0 or <0.

As a consequence, the feedback gains and the time delays must be chosen in such a way that

$$T_n = \frac{(2n\pi - \varphi)}{\Omega}, \tag{40}$$

where $n = 0, \pm 1, \pm 2, \ldots$. However, this choice is impossible, because it corresponds to a singularity in equation (36). A successful strategy is to choose the time delay near to the value in equation (40) in such a way that K does not increase excessively.

In Figure 3 the response amplitude of the approximate solution with no time delay (curve A) and with a time delay equal (case B) or near (case C) to the value in equation (40) has been shown. The parameter K is given by the condition (36) for cases A and C, while in case B it is equal to the value for case A. Comparison with the numerical solution has also been performed. The response is strongly suppressed when the time delay is near to the value in equation (40) and K is given by equation (36) (curve C).

In conclusion, the optimal choices for the time delay and the feedback gains are given by conditions (36) and (40), because the two-period quasiperiodic motion disappears and the peak amplitude is at its minimum value.

The strategy control can be summarized in the following two steps:

1. if the system response is simply periodic (condition (30) is not satisfied), the response amplitude can be reduced by choice (22) for the time delay, which is not dependent on the feedback gains;
2. if condition (30) is satisfied, a two-period quasiperiodic motion settles down and the only way to avoid it is to choose the feedback gains and the time delay in such a way as to satisfy condition (36) exactly and condition (40) approximately.

5. CONCLUSIONS

The state feedback control with a time delay has been studied for the primary resonance of a cantilever beam. By means of the asymptotic perturbation method and for small feedback gains, two slow-flow equations, governing the amplitude and phase of the approximate time response of the oscillator, have been derived. It has been found that the primary resonance of the oscillator with delay state feedback is qualitatively the same as that of the uncontrolled oscillator if the damping coefficient and the detuning parameter are redefined through the relations (14).

The external excitation–response and frequency-response curves have been compared with numerical solutions. Appropriate choices for the feedback gains and the time delay have been found in order to reduce the amplitude peak.

Moreover, the existence of quasiperiodic motion has been investigated. Using energy considerations, it has been demonstrated that no limit cycles (corresponding to a two-

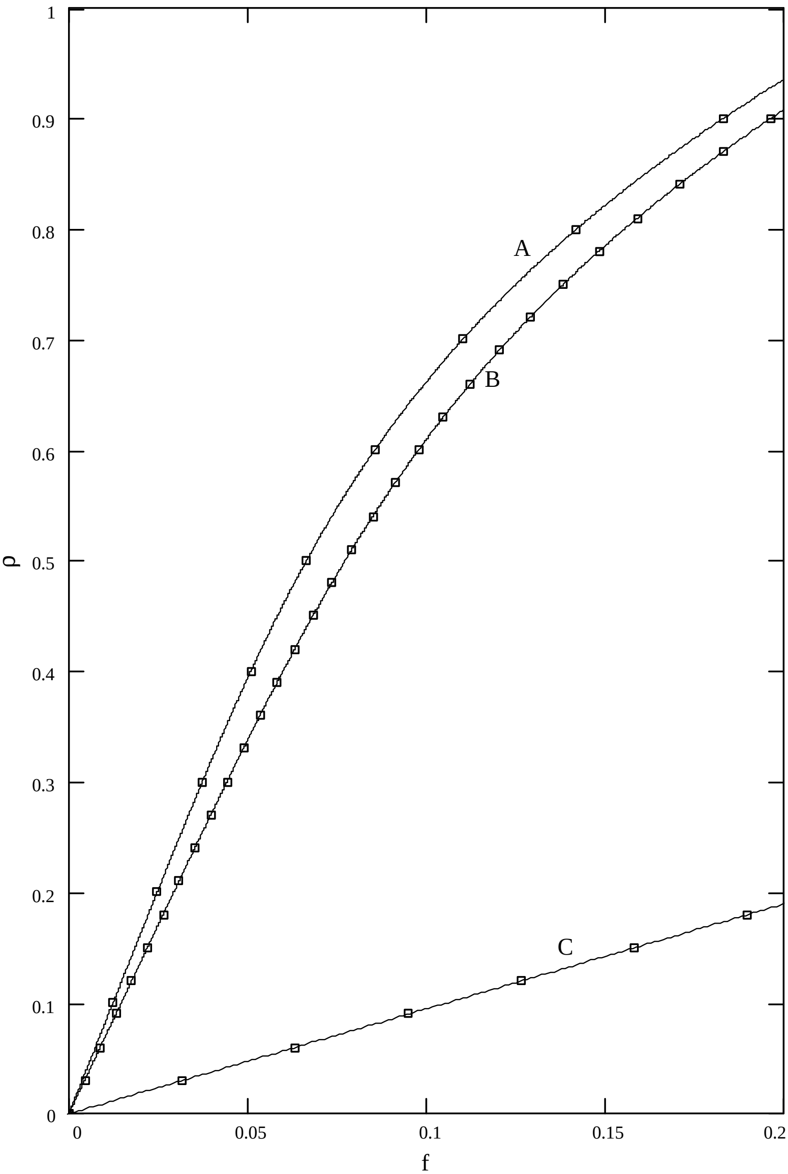

Figure 3. Parametric excitation (f)–response (ρ) curves for the controlled system with no time delay (curve A) and with time delay $T = -\pi/3$, (curve B) and $T = -0{\cdot}96\pi/3$ (curve C). The feedback gain K is given by equation (43) for the cases A ($K = 0{\cdot}023$) and C ($K = 0{\cdot}478$), while in the case B is $K = 0{\cdot}023$. Boxes stand for the numerical solution. ($\sigma = -0{\cdot}02$, $\alpha = 0{\cdot}1$, $\varphi = \pi/3$).

period modulated motion for the cantilever beam) exist for the slow-flow equations. Subsequently, the possibility of another type of quasiperiodic motion, in which the oscillator phase is unbounded, has been discovered.

In conclusion, appropriate choices for the time delay and the feedback gains can enhance the control performance, reduce the amplitude peak and suppress the quasiperiodic motion.

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